Chapter 2

Spectral Theory

In the first chapter, we show that the operators we studied are bounded between two Sobolev spaces. However, if we want to study the eigenvalues of an operator, we must assume that it is a linear operator on one Hilbert space. Unfortunately, if the operator has a positive order, it is not bounded on a Hilbert space. But for elliptic operator, we could prove that it is a closed unbounded operator on the canonical L^2-space. In this chapter, we will study the spectral theory, roughly speaking, the eigenvalues, of the closed unbounded operator.

2.1 Symmetry and Self-adjoint

2.1.1 Closed operator

As we discussed in Section 1.2.1, for a closed unbounded operator $T : \mathcal{H} \to \mathcal{H}$ on a Hilbert space $\mathcal{H}$, from the closed graph theorem: Theorem 1.2.11, T will only be defined on a linear subspace of $\mathcal{H}$. This subspace, which we denote by $D(T)$, is called the **domain** of the operator T (for the definition of the domain, we don't need T is closed).

Warning: To study an unbounded operator on a Hilbert space, we must first fix the domain and then see how it acts on that space.

Definition 2.1.1. The graph of the linear operator T is the set of pairs $\{\langle u, Tu\rangle \in \mathcal{H} \times \mathcal{H} : u \in D(T)\}$. The graph of T, denoted by $\Gamma(T)$, is a Hilbert space with inner product

$$(\langle u_1, v_1\rangle, \langle u_2, v_2\rangle) = (u_1, u_2) + (v_1, v_2). \tag{2.1.1}$$

The corresponding norm is denoted by $\|\cdot\|_\Gamma$. From (2.1.1),

$$\|\langle u, Tu\rangle\|_\Gamma^2 = \|u\|^2 + \|Tu\|^2. \tag{2.1.2}$$

Recall that in Definition 1.2.9, we say T is closed if for $u_k \in D(T)$, $u_k \to u$, $Tu_k \to v$, we have $u \in D(T)$ and $v = Tu$. The following proposition follows directly from Definition 1.2.9. In fact, in many literatures, this is the definition of the closed operator.

Proposition 2.1.2. *The linear operator T is closed if and only if $\Gamma(T)$ is a closed subspace of $\mathcal{H} \times \mathcal{H}$.*

Proof. Assume T is closed. If $\langle u_k, Tu_k \rangle$ converges in $\mathcal{H} \times \mathcal{H}$ with respect to the norm (2.1.1), since $\|\langle u_k, Tu_k \rangle\|_\Gamma^2 = \|u_k\|^2 + \|Tu_k\|^2$, we see that u_k and Tu_k converge. Let $u_k \to u$, $Tu_k \to v$, since T is closed, $u \in D(T)$ and $Tu = v$. Thus $\langle u_k, Tu_k \rangle \to \langle u, Tu \rangle \in \Gamma(T)$.

Assume that $\Gamma(T)$ is closed. If for $u_k \in D(T)$, $u_k \to u$, $Tu_k \to v$, since $\Gamma(T)$ is closed, the limit of $\langle u_k, Tu_k \rangle$ exists in $\Gamma(T)$, which is $\langle u, v \rangle$. Thus T is closed.

The proof of Proposition 2.1.2 is closed. □

Definition 2.1.3. Let T_1 and T_2 be linear operators on $\mathcal{H}$. We say T_2 is an extension of T_1, which we write $T_1 \subset T_2$, if $D(T_1) \subset D(T_2)$, i.e., $D(T_1) \subset D(T_2)$ and for any $u \in D(T_1)$, $T_2 u = T_1 u$.

Definition 2.1.4. An operator T is closable if it has a closed extension. The smallest closed extension, which exists obviously, is called the closure of T, denoted by $\overline{T}$.

In the followings, we will prove that the elliptic pesudodifferential operators on $\mathbb{R}^n$ or compact manifolds are closable. This is our main purpose to study closed operators here.

Proposition 2.1.5. *If M is compact, the elliptic pesudodifferential operator $P : \mathscr{C}^\infty(M, E) \to \mathscr{C}^\infty(M, E)$ of order $m > 0$ is closable. We will also denote the closure by P for the simplicity. In this case, $D(P) = \mathbf{H}^m(M, E)$.*

Proof. From the elliptic estimate (1.4.21), the norms $\|P \cdot\|_0 + \|\cdot\|_0$ and $\|\cdot\|_m$ are equivalent. From Definition 2.1.1, the norm $\|P \cdot\|_0 + \|\cdot\|$ is equivalent to the norm of the graph $\Gamma(P)$. Thus the closure of $\Gamma(P)$ is $\{(u, Pu) : u \in \mathbf{H}^m(M, E)\}$.

The proof of Proposition 2.1.5 is completed. □

In the same way, we have the corresponding result for $M = \mathbb{R}^n$.

Proposition 2.1.6. *The elliptic pesudodifferential operator $P : \mathscr{C}^\infty \to \mathscr{C}^\infty$ of order $m > 0$ is closable. We will also denote the closure by P for the simplicity. In this case, $D(P) = \mathbf{H}^m$.*

Remark 2.1.7. (1) If M is a general noncompact manifold, the case is complex. We need additional conditions to obtain the elliptic estimates. We will not discuss it in this note.

(2) If $m = 0$, by Proposition 1.3.2, P is bounded. If $m < 0$, by Rellich's theorem and Proposition 1.3.2, P is compact operator. They are easier to handle than closed operator.

(3) From Propositions 2.1.5 and 2.1.6, we see that in general, even if T is closed, $D(T)$ may be not a closed space.

Proposition 2.1.8. *If T is closable, then $\Gamma(\overline{T}) = \overline{\Gamma(T)}$. Thus we could obtain the closure of T be taking the closure of $\Gamma(T)$.*

Proof. Suppose that S is a closed extension of T. Then $\overline{\Gamma(T)} \subset \overline{\Gamma(S)} = \Gamma(S)$. Thus if $\langle 0, v\rangle \in \overline{\Gamma(T)}$, $v = 0$. Let $A = \{u : \exists v, s.t. \langle u, v\rangle \in \overline{\Gamma(T)}\}$. Thus for $u \in A$, there exists unique $v \in \mathcal{H}$ such that $\langle u, v\rangle \in \overline{\Gamma(T)}$. Define R by $Ru = v$. Then $\Gamma(R) = \overline{\Gamma(T)}$. So R is a closed extension of T. Since $R \subset S$ for any closed extension S, we have $\overline{T} = R$.

The proof of Proposition 2.1.8 is completed. □

Remark that for general linear operator T, the closure of $\Gamma(T)$ may not be a graph of an operator.

Proposition 2.1.9. *A linear operator T is closable if and only if for $u_k \in D(T)$, $u_k \to 0$, $Tu_k \to v$, we have $v = 0$.*

Proof. If $u_k \in D(T)$, $u_k \to 0$, $Tu_k \to v$, then $\langle 0, v\rangle \in \overline{\Gamma(T)}$. Since T is closable, by Proposition 2.1.8, $\langle 0, v\rangle \in \Gamma(\overline{T})$. Thus $v = 0$.

For the other direction, if $\langle u, v_1\rangle, \langle u, v_2\rangle \in \overline{\Gamma(T)}$, then there exist $u_k \to u$, $u_k' \to u$ such that $Tu_k \to v_1$ and $Tu_k' \to v_2$. Note that $u_k - u_k' \in D(T)$. Thus $u_k - u_k' \to 0$ and $T(u_k - u_k') \to v_1 - v_2$. So $v_1 = v_2$. Thus we could define a operator R such that for any $\langle u, v\rangle \in \overline{\Gamma(T)}$, $v = Ru$. Since $\Gamma(R) = \overline{\Gamma(T)}$ is closed, by Proposition 2.1.2, R is a closed operator, which is a closed extension of T.

The proof of Proposition 2.1.9 is completed. □

Now we summarize some properties of the closed operator.

Proposition 2.1.10. *(1) If T is a $1-1$ closed operator, then T^{-1} is closed.*

(2) If T is closed, then $\mathrm{Ker}(T)$ is a closed space.

(3) If T is closed, $D(T) = H$, then T is bounded.

Proof. (1) Rotate $\Gamma(T)$ and use Proposition 2.1.2.

(2) If $u_k \in \mathrm{Ker}(T)$, $u_k \to u$, then $Tu_k \equiv 0$. Since T is closed, $Tu = 0$.

(3) It is the closed graph theorem (Theorem 1.2.11).

The proof of Proposition 2.1.10 is completed. □

2.1.2 Symmetric and self-adjoint

Now we study the adjoint of a linear operator. In order to get a well-defined adjoint operator, we need the condition that the domain is dense in $\mathcal{H}$.

Definition 2.1.11. We say T is densely defined if $D(T)$ is dense in $\mathcal{H}$.

From the closed graph theorem, for unbounded closed operator, $D(T) \neq \mathcal{H}$. We can always assume that $D(T)$ is dense in $\mathcal{H}$. If not, we consider the Hilbert space $\overline{D(T)}$.

Remark that since the space of the smooth functions is dense in the L^2-space, all elliptic operators and pseudodifferential operators are densely defined.

Definition 2.1.12. Let T be a densely defined linear operator on $\mathcal{H}$. Let $D(T^*)$ be the set of $v \in \mathcal{H}$ for which there exists $w \in \mathcal{H}$ such that for any $u \in D(T)$,

$$(Tu, v) = (u, w). \tag{2.1.3}$$

For each such $u \in D(T^*)$, we define $T^*v = w$. The operator T^* is called the adjoint of T.

Obviously, T^* is linear. Since T is densely defined, T^* is well-defined: if for any $u \in D(T)$, $(u, w) = (u, w')$, then $w = w'$. In general $D(T^*)$ may not be dense.

Proposition 2.1.13. *Let T, S be a densely defined linear operators on $\mathcal{H}$.*

(1) The element $v \in D(T^*)$ if and only if there exists a constant $C_v > 0$ such that for any $u \in D(T)$, $(Tu, v) \leq C_v\|u\|$.

(2) The adjoint T^* is closed.

(3) If $S \subset T$, then $T^* \subset S^*$.

(4) T is closable if and only if $D(T^*)$ is dense, in which case $\overline{T} = T^{**}$.

(5) If T is closable, then $(\overline{T})^* = T^*$.

(6) T is closed if and only if T^* is densely defined and $T = T^{**}$.

Proof. (1) If $u \in D(T^*)$, by (2.1.3), $(Tu, v) \leq \|w\|\|u\|$. If for any $u \in D(T)$, $(Tu, v) \leq C_v\|u\|$, then by Riesz representation theorem[1], since (Tu, v) is bounded linear on u, there exists $w \in \mathcal{H}$ such that $(Tu, v) = (u, w)$.

(2) We define a operator V on $\mathcal{H}\times\mathcal{H}$ by $V\langle u, v\rangle = \langle -v, u\rangle$. Then $V(\Gamma(T))$ is a linear space. We claim that

$$\Gamma(T^*) = (V(\Gamma(T)))^{\perp}, \tag{2.1.4}$$

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[1] Theorem 2.2.1 in ”Functional analysis” by Zhang

where $\cdot^{\perp}$ is the complement with respect to the inner product (2.1.1). In fact, if $u \in D(T^*)$, for any $\langle w, Tw\rangle \in \Gamma(T)$,

$$\begin{aligned}(\langle u, T^*u\rangle, V\langle w, Tw\rangle) &= (\langle u, T^*u\rangle, \langle -Tw, w\rangle)\\ &= -(u, Tw) + (T^*u, w) = 0. \end{aligned} \tag{2.1.5}$$

Thus $\Gamma(T^*) \subset (V(\Gamma(T)))^{\perp}$. On the other hand, if $\langle u, v\rangle \in (V(\Gamma(T)))^{\perp}$, then for any $\langle w, Tw\rangle \in \Gamma(T)$,

$$0 = (\langle u, v\rangle, V\langle w, Tw\rangle) = (\langle u, v\rangle, \langle -Tw, w\rangle) = -(u, Tw) + (v, w). \tag{2.1.6}$$

Thus $\langle u, v\rangle \in \Gamma(T^*)$. Then (2.1.4) holds.

Note that $(V(\Gamma(T)))^{\perp}$ is a closed subspace. So is $\Gamma(T^*)$. From Proposition 2.1.2, T^* is closed.

(3) Since $\Gamma(S) \subset \Gamma(T)$, $V(\Gamma(S)) \subset V(\Gamma(T))$. Thus (3) follows from (2.1.4).

(4) We claim that for any subspace A in $\mathcal{H} \times \mathcal{H}$,

$$V(A^{\perp}) = V(A)^{\perp}. \tag{2.1.7}$$

This follows from

$$\begin{aligned}(\langle u, v\rangle, V\langle w, t\rangle) &= (\langle u, v\rangle, \langle -t, w\rangle) = -(u, t) + (v, w)\\ &= (\langle v, -u\rangle, \langle w, t\rangle) = (V\langle u, v\rangle, \langle w, t\rangle), \end{aligned} \tag{2.1.8}$$

for any $(w, t) \in A$.

If $D(T^*)$ is dense, by (2), T^{**} is well-defined and closed. By (2.1.4) and (2.1.7),

$$\begin{aligned}\Gamma(T^{**}) = (V(\Gamma(T^*)))^{\perp} &= \left(V\left(V(\Gamma(T))^{\perp}\right)\right)^{\perp} = \left(V^2\left(\Gamma(T)^{\perp}\right)\right)^{\perp}\\ &= \left(\Gamma(T)^{\perp}\right)^{\perp} = \overline{\Gamma(T)}. \end{aligned} \tag{2.1.9}$$

From Proposition 2.1.8, T is closable and $\overline{T} = T^{**}$.

If $D(T^*)$ is not dense, take $v \in D(T^*)^{\perp}$, $v \neq 0$. Then $\langle v, 0\rangle \in \Gamma(T^*)^{\perp}$. So $V(\Gamma(T^*))^{\perp}$ is not a graph of an operator. Since by (2.1.9), $\overline{\Gamma(T)} = V(\Gamma(T^*))^{\perp}$, we see that T is not closable.

(5) If T is closable, from (2) and (4),

$$T^* = \overline{T^*} = T^{***} = \overline{T}^*. \tag{2.1.10}$$

(6) follows directly from (4).

The proof of Proposition 2.1.13 is completed. □

Definition 2.1.14. Let T be a densely defined linear operator on $\mathcal{H}$. If $T \subset T^*$, we say T is **symmetric**. If $T = T^*$, we say T is **self-adjoint**. If T is closable and $\overline{T}$ is self-adjoint, we say T is **essentially self-adjoint**.

If T is symmetric, $D(T) \subset D(T^*)$. Thus T^* is densely defined. By Proposition 2.1.13 (2) and (4), we have

Proposition 2.1.15. *(1) Symmetric operator is closable.*
(2) Self-adjoint operator is closed.

The following proposition is obvious.

Proposition 2.1.16. *A densely defined linear operator T is symmetric if and only if for any $u, v \in D(T)$,*

$$(Tu, v) = (u, Tv). \tag{2.1.11}$$

Proposition 2.1.17. *A densely defined operator T is essentially self-adjoint if and only if $T \subset T^{**} = T^*$.*

Proof. Since T is closable. By Proposition 2.1.13 (4), $\overline{T} = T^{**}$.

If T is essentially self-adjoint, $\overline{T}$ is self-adjoint. By Proposition 2.1.13 (5), $T^{**} = \overline{T} = (\overline{T})^* = T^*$.

If $T \subset T^{**} = T^*$, $(\overline{T})^* = T^* = T^{**} = \overline{T}$. Thus $\overline{T}$ is self-adjoint.

The proof of Proposition 2.1.17 is completed. □

Proposition 2.1.18. *Let T be a densely defined symmetric operator. The following statements are equivalent.*
(1) T is self-adjoint.
(2) $D(T) = D(T^*)$.
(3) $T = T^{**} = T^*$.

Proof. Obvious. □

Remark 2.1.19. We compare the properties of closed, essentially self-adjoint and self-adjoint for a densely defined symmetric operator. Let T be a densely defined symmetric operator. Then

- T is closed $\Longleftrightarrow T = T^{**} \subset T^*$;
- T is essentially self-adjoint $\Longleftrightarrow T \subset T^{**} = T^*$;
- T is self-adjoint $\Longleftrightarrow T = T^{**} = T^*$.

For symmetric operator T on $\mathcal{H}$, for $\mu > 0$, we have

$$\|(T \pm i\mu \,\mathrm{Id})x\|^2 = ((T \pm i\mu \,\mathrm{Id})x, (T \pm i\mu \,\mathrm{Id})x) = \mu^2\|x\|^2 + \|Tx\|^2. \quad (2.1.12)$$

So for any $\mu > 0$, we have

$$\mathrm{Ker}(T \pm i\mu \,\mathrm{Id}) = 0. \quad (2.1.13)$$

Lemma 2.1.20. *If T is a closed symmetric operator, then* $\mathrm{Im}(T \pm i\mu \,\mathrm{Id})$ *is closed.*

Proof. If $(T + i\mu \,\mathrm{Id})x_k \to y$, since $\|(T + i\mu \,\mathrm{Id})(x_k - x_j)\|^2 = \mu^2\|x_k - x_j\|^2 + \|T(x_k - x_j)\|^2$, $x_k \to x$, $Tx_k \to w$. Since T is closed, $w = Tx$. Thus $y = w + i\mu x = (T + i\mu \,\mathrm{Id})x$. The proof for $T - i\mu \,\mathrm{Id}$ is the same.

The proof of Lemma 2.1.20 is completed. □

Lemma 2.1.21. *Let T be a densely defined symmetric operator. Then for any $\mu > 0$,*

$$\mathrm{Ker}(T^* \pm i\mu \,\mathrm{Id}) = \mathrm{Im}(T \mp i\mu \,\mathrm{Id})^{\perp}. \quad (2.1.14)$$

Proof. If $v \in \mathrm{Ker}(T^* + i\mu \,\mathrm{Id})$, $v \in D(T^*)$. For any $u \in D(T)$,

$$((T - i\mu \,\mathrm{Id})u, v) = (u, (T^* + i\mu \,\mathrm{Id})v) = 0. \quad (2.1.15)$$

Thus $v \in \mathrm{Im}(T - i\mu \,\mathrm{Id})^{\perp}$.

If $v \in \mathrm{Im}(T - i\mu \,\mathrm{Id})^{\perp}$, also as in (2.1.15), $v \in \mathrm{Ker}(T^* + i\mu \,\mathrm{Id})$.

The proof for $T^* - i\mu \,\mathrm{Id}$ is the same.

The proof of Lemma 2.1.21 is completed. □

Proposition 2.1.22. *Let T be a densely defined symmetric operator. Then the following statements are equivalent:*

(1) T is self-adjoint;

(2) T is closed and there exists $\mu > 0$ such that $\mathrm{Ker}(T^* \pm i\mu \,\mathrm{Id}) = 0$*;*

(3) there exists $\mu > 0$ such that $\mathrm{Im}(T \mp i\mu \,\mathrm{Id}) = H$*.*

Proof. For (1) $\Longrightarrow$ (2), we use Proposition 2.1.15 and (2.1.13).

For (2) $\Longrightarrow$ (3), we use Lemmas 2.1.20 and 2.1.21.

For (3) $\Longrightarrow$ (1), by Lemma 2.1.21, $\mathrm{Ker}(T^* \pm i\mu \,\mathrm{Id}) = 0$. For any $v \in D(T^*)$, there exists $u \in D(T)$ such that $(T^* \mp i\mu \,\mathrm{Id})v = (T \mp i\mu \,\mathrm{Id})u$. Since $T \subset T^*$, we have $(T^* \mp i\mu \,\mathrm{Id})(v - u) = 0$. Thus $v = u \in D(T)$. So $D(T) = D(T^*)$. By Proposition 2.1.18, T is self-adjoint.

The proof of Proposition 2.1.22 is completed. □

Proposition 2.1.23. *Let T be a densely defined symmetric operator. Then the following statements are equivalent:*

(1) T is essentially self-adjoint;

(2) there exists $\mu > 0$ such that $\mathrm{Ker}(T^* \pm i\mu\,\mathrm{Id}) = 0$;

(3) there exists $\mu > 0$ such that $\overline{\mathrm{Im}(T \mp i\mu\,\mathrm{Id})} = H$.

Proof. For (1) $\Longrightarrow$ (2), by Proposition 2.1.17, T^* is self-adjoint. From (2.1.13), we get (2).

(2)$\Leftrightarrow$(3) is obvious.

For (3) $\Longrightarrow$ (1), for any $v \in D(T^*)$, there exists $\{u_n\} \subset \mathcal{H}$ such that

$$\lim_{n\to\infty}(T \mp i\mu\,\mathrm{Id})u_n = (T^* \mp i\mu\,\mathrm{Id})v. \tag{2.1.16}$$

It implies that $\lim_{n\to\infty}(T^* \mp i\mu\,\mathrm{Id})(u_n - v) = 0$. Thus $\lim_{n\to\infty}\|(T^* \mp i\mu\,\mathrm{Id})(u_n - v)\|^2 = \lim_{n\to\infty}\mu^2\|u_n - v\|^2 + \lim_{n\to\infty}\|T^*(u_n - v)\|^2 = 0$. So $u_n \to v$, $Tu_n \to T^*v$. Thus $(v, T^*v) \in \overline{\Gamma(T)} = \Gamma(\overline{T})$. So $v \in D(\overline{T}) = D(T^{**})$, which means that $T^* \subset T^{**}$.

On the other hand, from Proposition 2.1.13, $T^{**} \subset T^*$. So $T^* = T^{**}$. From Proposition 2.1.17, T is essentially self-adjoint.

The proof of Proposition 2.1.23 is completed. □

Theorem 2.1.24 (von Neumann)**.** *Let T be a densely defined closed symmetric operator. We have*

$$D(T^*) = D(T) \oplus \mathrm{Ker}(T^* - i\,\mathrm{Id}) \oplus \mathrm{Ker}(T^* + i\,\mathrm{Id}). \tag{2.1.17}$$

Moreover, for $u = u_0 + u_+ + u_- \in D(T^*)$ with decomposition as in (2.1.17), we have $T^*u = Tu_0 + iu_+ - iu_-$.

Proof. Denote by $D_\pm := \mathrm{Ker}(T^* \mp i\,\mathrm{Id})$.

We first claim that $D(T)$, D_+ and D_- are linear independent. In fact, if $u_0 + u_+ + u_- = 0$, where $u_0 \in D(T)$, $u_\pm \in D_\pm$, since $(T^* - i\,\mathrm{Id})u_- = -2iu_-$, we have $(T - i\,\mathrm{Id})u_0 = 2iu_-$. But from Lemma 2.1.21, we have $u_- \in \mathrm{Ker}(T^* + i\,\mathrm{Id}) = \mathrm{Im}(T - i\,\mathrm{Id})^\perp$. Thus $u_- = 0$. Similarly, we get $u_+ = 0$. Therefore, $D(T)$, D_+ and D_- are linear independent.

Obviously, $D(T) \oplus D_+ \oplus D_- \subset D(T^*)$. We claim that $D(T^*) \subset D(T) \oplus D_+ \oplus D_-$. In fact, from Lemmas 2.1.20 and 2.1.21, $\mathcal{H} = \mathrm{Im}(T - i\,\mathrm{Id}) \oplus D_-$. For any $u \in D(T^*)$, $v = (T^* - i\,\mathrm{Id})u$ has decomposition $v = v_1 + v_2$, where $v_1 \in \mathrm{Im}(T - i\,\mathrm{Id})$, $v_2 \in D_-$. Take u_0 such that $(T - i\,\mathrm{Id})u_0 = v_1$. Let $u_- = -(2i)^{-1}v_2$. Since $T^*u_- = -iu_-$, $(T^* - i\,\mathrm{Id})u_- = -2iu_- = v_2$. Then $(T^* - i\,\mathrm{Id})(u_0 + u_-) = v$. So $(T^* - i\,\mathrm{Id})(u - u_0 - u_-) = 0$. Let $u_+ = u - u_0 - u_- \in D_+$. We obtain the claim.

The proof of Theorem 2.1.24 is completed. □

2.1.3 Friedrichs extension

Proposition 2.1.25. *Essentially self-adjoint operator has unique self-adjoint extension.*

Proof. Let T be a essentially self-adjoint operator, then $\bar{T} = T^{**} = T^*$ is a self-adjoint extension. If T_0 be another self-adjoint extension, then $T^*\bar{T} \subset T_0$. Since $T_0 = T_0^* \subset T^{**} = T^*$, we have $\bar{T} = T_0$.

The proof of Proposition 2.1.25 is completed. □

In general, a symmetric operator may have many self-adjoint extensions. In this subsection, we introduce the Friedrich extension.

Definition 2.1.26. Let V be a dense linear subset of $\mathcal{H}$. Let $a : V \times V \to \mathbb{C}$ be a sesquilinear form, i.e., for any $b, c \in \mathbb{C}$, $u, v \in V$,

$$a(bu, cv) = b\bar{c} \cdot a(u, v), \tag{2.1.18}$$

such that a is positive definite, i.e., there exists $\alpha > 0$, such that for any $v \in V$,

$$a(v, v) \geq \alpha \|v\|^2. \tag{2.1.19}$$

Lemma 2.1.27. *If a is positive definite, a is symmetric, i.e., for any $u, v \in V$,*

$$a(u, v) = \overline{a(v, u)}. \tag{2.1.20}$$

Proof. For any $\lambda \in \mathbb{C}$,

$$0 \leq a(\lambda u + v, \lambda u + v) = |\lambda|^2 a(u, u) + \lambda a(u, v) + \bar{\lambda} a(v, u) + a(v, v). \tag{2.1.21}$$

Thus $\lambda a(u, v) + \bar{\lambda} a(v, u) \in \mathbb{R}$. It is equivalent to

$$\lambda a(u, v) - \overline{\bar{\lambda} a(v, u)} = \lambda(a(u, v) - \overline{a(v, u)}) \in \mathbb{R}. \tag{2.1.22}$$

Since $\lambda \in \mathbb{C}$ is arbitrary taken, we have $a(u, v) = \overline{a(v, u)}$.

The proof of Lemma 2.1.27 is completed. □

Lemma 2.1.28 (Schwarz inequality)**.** *If a is positive definite, for any $u, v \in V$,*

$$|a(u, v)|^2 \leq a(u, u) a(v, v). \tag{2.1.23}$$

Proof. From (2.1.21), for any $\lambda \in \mathbb{C}$,

$$|\lambda|^2 a(u,u) + 2\mathrm{Re}(\lambda a(u,v)) + a(v,v) \geq 0. \tag{2.1.24}$$

If $a(u,v) = re^{i\theta}$, we take $\lambda = te^{-i\theta}$. Then (2.1.24) is

$$a(u,u)t^2 + 2rt + a(v,v) \geq 0 \tag{2.1.25}$$

for any $t \in \mathbb{R}$. Thus

$$r^2 \leq a(u,u)a(v,v). \tag{2.1.26}$$

The proof of Lemma 2.1.28 is completed. □

From Lemmas 2.1.27 and 2.1.28, we see that $a(\cdot,\cdot)$ is an inner product on V. It induces a norm on V:

$$\|v\|_a^2 = a(v,v)^{1/2} \tag{2.1.27}$$

for any $v \in V$. From (2.1.19), this norm is stronger than the normal norm of the Hilbert space.

Definition 2.1.29. Let a be a positive definite sesquilinear form. We denote by $D(a) := V$ the domain of a. If $D(a)$ is complete with respect to the norm $\|\cdot\|_a$, we say a is closed. In this case, $D(a)$ is a Hilbert space with respect to the norm $\|\cdot\|_a$.

For a symmetric operator T, since for any $u \in D(T)$, $(Tu,u) = (u,Tu) = \overline{(Tu,u)}$, we have $(Tu,u) \in \mathbb{R}$. We say a self-adjoint operator T is positive definite if there exists $\alpha > 0$ such that for any $u \in D(T)$, $(Tu,u) \geq \alpha\|u\|^2$.

Proposition 2.1.30. *Let a be a closed positive definite sesquilinear form with domain V. Then there exists unique positive definite self-adjoint operator T, such that $D(T) \subset V$ and for any $u \in D(T)$, $v \in V$,*

$$(v,Tu) = a(v,u). \tag{2.1.28}$$

Proof. Set

$$D(T) := \{u \in V : \exists C_u > 0,\ s.t.\ \forall v \in V, |a(v,u)| \leq C_u\|v\|\}. \tag{2.1.29}$$

By Riesz representation theorem, there exists unique $u^* \in \mathcal{H}$ such that

$$a(v,u) = (v,u^*). \tag{2.1.30}$$

We define $Tu = u^*$ for any $u \in D(T)$.

Obviously T is linear. We claim that T is densely defined. Since $\|\cdot\|_a$ is stronger than $\|\cdot\|$, for this claim, we only need to prove that $D(T)$ is dense in V with respect to $\|\cdot\|_a$, which is equivalent to prove that if $v_0 \in V$ and for any $u \in D(T)$, $a(v_0, u) = 0$, then $v_0 = 0$. In fact, for any $w \in \mathcal{H}$,

$$|(v, w)| \leq \|v\|\|w\| \leq \frac{1}{\sqrt{\alpha}}\|w\|\|v\|_a. \tag{2.1.31}$$

From Riesz representation theorem with respect to the inner product a, there exists $u_0 \in V$,

$$(v, w) = a(v, u_0). \tag{2.1.32}$$

Thus

$$\mathrm{Im}(T) = \mathcal{H}. \tag{2.1.33}$$

So if $0 = a(v_0, u) = (v_0, Tu)$ for any $u \in D(T)$, $v_0 = 0$. We obtain the claim.

For any $u, v \in D(T)$,

$$(v, Tu) = a(v, u) = \overline{a(u, v)} = \overline{(u, Tv)} = (Tv, u). \tag{2.1.34}$$

So T is symmetric.

Now we prove that T is self-adjoint. We only need to prove that $D(T^*) \subset D(T)$. If $u \in D(T^*)$, then there exists $u^* \in \mathcal{H}$ such that for any $v \in D(T)$,

$$(u^*, v) = (u, Tv). \tag{2.1.35}$$

From (2.1.33), there exists $w \in D(T)$ such that $u^* = Tw$. Thus $(w, Tv) = (Tw, v) = (u^*, v) = (u, Tv)$ for any $v \in D(T)$. From (2.1.33) again, we have $u = w \in D(T)$.

Since a is positive definite, T is positive definite.

At last, we prove the uniqueness. If T' be another self-adjoint operator satisfying all conditions. For any $u \in D(T')$, $T'u \in \mathcal{H} = \mathrm{Im}(T)$. Thus there exists $w \in D(T)$ such that $T'u = Tw$. Thus for any $v \in V$,

$$a(v, u) = (v, T'u) = (v, Tw) = a(v, w). \tag{2.1.36}$$

Since a is positive definite, by taking $v = u - w$, we get $u = w \in D(T)$. So $T' \subset T$. In the same way, $T \subset T'$. Thus $T = T'$.

The proof of Proposition 2.1.30 is completed. □

Definition 2.1.31. Let T be a symmetric operator. If there exists $c \in \mathbb{R}$ such that for any $u \in D(T)$,

$$(u, Tu) \geq c(u, u), \tag{2.1.37}$$

we say T is bounded from below. We also write $T \geq c$.

Theorem 2.1.32 (Friedrichs extension)**.** *Let T be a symmetric operator. If $T \geq -M$, then we can construct a self-adjoint extension $\hat{T}$ of T, called the* ***Friedrich extension****, such that $\hat{T} \geq -M$.*

Proof. We assume that $T \geq 1$ first. In this case,

$$(u, Tu) \geq \|u\|^2 \tag{2.1.38}$$

for any $u \in D(T)$. Let

$$a(v, u) = (v, Tu) \tag{2.1.39}$$

for any $u, v \in D(T)$. Let V be the closure of $D(T)$ with respect to $\| \cdot \|_a$. Then a could be extended on $V \times V$, denoted by $\hat{a}(\cdot, \cdot)$. Then $\hat{a}$ is a positive definite sesquilinear form.

We claim that $V \subset \mathcal{H}$. In this case, $\hat{a}$ is closed and we can use Proposition 2.1.30. In fact, let $i : D(T) \to \mathcal{H}$ be the embedding. For any $v \in V$, there exists $\{u_n\} \subset D(T)$ such that $\lim_{n\to\infty} u_n = v$ with respect to $\| \cdot \|_a$. Thus $\{u_n\}$ is a Cauchy sequence with respect to $\| \cdot \|_a$. From (2.1.38), $\{u_n\}$ is also a Cauchy sequence with respect to $\| \cdot \|$. Thus there exists $v^* \in \mathcal{H}$ such that $\lim_{n\to\infty} u_n = v$ with respect to $\| \cdot \|$. So we can embed V into $\mathcal{H}$ by identifying v with v^*. This process is similar as we did in the proof of the Sobolev embedding theorem.

Now $\hat{a}$ is a closed positive definite sesquilinear form. From Proposition 2.1.30, there exists a unique positive definite self-adjoint operator $\hat{T}$ such that

$$(v, \hat{T}u) = \hat{a}(v, u) \tag{2.1.40}$$

for any $u \in D(\hat{T})$ and $v \in V$. Easy to see that $\hat{T} \geq 1$.

For $u \in D(T)$, from (2.1.39), for any $v \in D(T)$,

$$|a(v, u)| = |(v, Tu)| \leq \|Tu\| \|v\|. \tag{2.1.41}$$

This inequality also holds for any $v \in V$. In fact, if $u_n \to v$ with respect to $\| \cdot \|_a$, $u_n \in D(T)$, then from (2.1.38), $u_n \to v$ with respect to $\| \cdot \|$. We could take limit in (2.1.41). By (2.1.29), (2.1.41) holds for any $v \in V$ implies that $u \in D(\hat{T})$. Since

$$(v, Tu) = a(v, u) = (v, \hat{T}u) \tag{2.1.42}$$

for any $v \in D(T)$, we have $Tu = \hat{T}u$. So $T \subset \hat{T}$, $\hat{T}$ is the self-adjoint extension of T.

In general case, if $T \geq -M$, let $T' = T + (M+1)\,\mathrm{Id}$. Then $T' \geq 1$. So T' has a self-adjoint extension $\hat{T'}$ and $\hat{T'} \geq 1$. Then $\hat{T} := \hat{T'} - (M+1)\,\mathrm{Id}$ is a self-adjoint extension of T and $\hat{T} \geq -M$.

The proof of Theorem 2.1.32 is completed. □

From Definition 1.4.12, the generalized Laplacian H is symmetric and bounded from below. From Theorem 2.1.32, it has Friedrich extension. (This is a reason why we need Q is bounded from below in the definition of the generalized Laplacian.) In fact, it is its unique self-adjoint extension.

Corollary 2.1.33. *The symmetric generalized Laplacian H on a compact manifold in Definition 1.4.12 is essentially self-adjoint. We also denote the unique self-adjoint extension by H.*

Proof. We only need to prove that the Friedrich extension $\hat{H} = \overline{H}$. Assume that $H \geq -M$. Then the norm $\|\cdot\|_a$ we considered in the proof of Theorem 2.1.32 is $(\cdot, H\cdot) + (1+M)\|\cdot\|$, which is equivalent to $\|\cdot\|_1$ from the Garding inequality, Theorem 1.4.13. Thus the dense subspace V in the proof of Theorem 2.1.32 is $\mathbf{H}^1$. From (2.1.29), if $u \in D(\hat{H})$, there exists $C_u > 0$, such that for any $v \in \mathbf{H}^1$, $|(v, \hat{H}u)| \leq C_u\|v\|$, which means that

$$\|\hat{H}u\| = \sup_{v \in \mathscr{C}^\infty} \frac{(v, \hat{H}u)}{\|v\|} \leq C_u < \infty. \tag{2.1.43}$$

So $\hat{H}u \in \mathbf{H}^0(E) = L^2(M, E)$. From elliptic estimate, $\|u\|_2 \leq C(\|\hat{H}u\| + \|u\|) < +\infty$. Thus $u \in \mathbf{H}^2(E)$. Therefore $D(\hat{H}) \subset \mathbf{H}^2(E)$.

From Proposition 2.1.5, $D(\overline{H}) = \mathbf{H}^2(E)$. Since $\overline{H}$ is the smallest closed extension of H, we have $\mathbf{H}^2(E) = D(\overline{H}) \subset D(\hat{H})$. So $D(\overline{H}) = D(\hat{H}) = \mathbf{H}^2(E)$. Since $\hat{H}$ and $\overline{H}$ are completed with respect to the equivalent norms, we have $\hat{H} = \overline{H}$.

The proof of Corollary 2.1.33 is completed. □

2.1.4 Perturbation

Definition 2.1.34. Let A and B be densely defined operators on $\mathcal{H}$. If $D(A) \subset D(B)$ and there exist $a, b > 0$ such that for any $u \in D(A)$,

$$\|Bu\| \leq a\|Au\| + b\|u\|, \tag{2.1.44}$$

we say B is A-bounded with bound a.

Theorem 2.1.35. *Let A and B be densely defined operators on $\mathcal{H}$ and B is A-bounded with bound $a < 1$. Then $A + B$ on $D(A)$ is closable if and only if A is closable. In this case,*

$$D(\overline{A+B}) = D(\overline{A}). \tag{2.1.45}$$

Proof. From (2.1.44), for any $u \in D(A)$,

$$\begin{aligned}(1-a)\|Au\| - b\|u\| \le \|Au\| - \|Bu\| \le \|(A+B)u\| \\ \le \|Au\| + \|Bu\| \le (1+a)\|Au\| + b\|u\|. \quad (2.1.46)\end{aligned}$$

So if $a < 1$, for $\{u_n\} \subset D(A)$ converges, $\{Au_n\}$ is a Cauchy sequence if and only if $\{(A+B)u_n\}$ is a Cauchy sequence. If $u_n \to 0$, $Au_n \to 0$ if and only if $(A+B)u_n \to 0$. From Proposition 2.1.9, $A+B$ on $D(A)$ is closable if and only if A is closable.

If A is closable, $A+B$ is closable. If $u \in D(\overline{A})$, there exists $\{u_n\} \subset D(A)$, $u_n \to u$ and Au_n convergences. From (2.1.46), $(A+B)u_n$ also convergences. So $u \in D(\overline{A+B})$. In the same way, $D(\overline{A+B}) \subset D(\overline{A})$. So $D(\overline{A+B}) = D(\overline{A})$.

The proof of Theorem 2.1.35 is completed. □

Corollary 2.1.36. *If B is A-bounded with bound $a < 1$. Then $A+B$ is closed if and only if A is closed.*

Proof. Obvious. □

Theorem 2.1.37 (Kato-Rellich Theorem)**.** *If A is self-adjoint, B is symmetric and B is A-bounded with bound $a < 1$, then $A+B$ is self-adjoint.*

Proof. From Proposition 2.1.22, we only need to prove that there exists $\mu_0 > 0$ such that $\operatorname{Im}(A+B \pm i\mu_0 \operatorname{Id}) = \mathcal{H}$.

For any $\mu > 0$, from (2.1.13), $\operatorname{Ker}(A \pm i\mu \operatorname{Id}) = 0$. Since A is self-adjoint, by Proposition 2.1.22, $\operatorname{Im}(A \pm i\mu \operatorname{Id}) = \mathcal{H}$. Then $(A \pm i\mu \operatorname{Id})^{-1}$ is well defined on $\mathcal{H}$. For any $u \in \mathcal{H}$, taking $x = (A \pm i\mu \operatorname{Id})^{-1}u$ in (2.1.12), we get

$$\|A(A \pm i\mu \operatorname{Id})^{-1}\| \le 1, \quad \|(A \pm i\mu \operatorname{Id})^{-1}\| \le \frac{1}{\mu}. \quad (2.1.47)$$

Note that

$$A + B \pm i\mu \operatorname{Id} = (B(A \pm i\mu \operatorname{Id})^{-1} - \operatorname{Id})(A \pm i\mu \operatorname{Id}). \quad (2.1.48)$$

From Definition 2.1.34 and (2.1.47),

$$\|B(A \pm i\mu) \operatorname{Id})^{-1}\| \le a\,\|A(A \pm i\mu \operatorname{Id})^{-1}\| + b\,\|(A \pm i\mu \operatorname{Id})^{-1}\| \le a + \frac{b}{\mu}. \quad (2.1.49)$$

Since $a < 1$, we could take $\mu_0 > 0$ large enough such that $a + b\mu_0^{-1} < 1$. In this case, $(B(A \pm i\mu \operatorname{Id})^{-1} - \operatorname{Id})$ is invertible. From (2.1.48), since $\operatorname{Im}(A \pm i\mu_0 \operatorname{Id}) = \mathcal{H}$, we have $\operatorname{Im}(A+B \pm i\mu_0 \operatorname{Id}) = \mathcal{H}$.

The proof of Theorem 2.1.37 is completed. □

2.1.5 Hodge decomposition

Proposition 2.1.38. *Let $P : \mathscr{C}^\infty(M, E) \to \mathscr{C}^\infty(M, E)$ be an essentially self-adjoint elliptic differential operator over a compact Riemannian manifold of order $m > 0$. Then there is an L^2-orthogonal direct sum decomposition*

$$\mathscr{C}^\infty(M, E) = \operatorname{Ker} P \oplus \operatorname{Im} P. \tag{2.1.50}$$

Proof. With respect to the L^2-decomposition $L^2(M, E) = \operatorname{Ker} P \oplus (\operatorname{Ker} P)^\perp$, for any $u \in \mathscr{C}^\infty(M, E)$, we can write $u = u_0 + u_1$ and $Pu_0 = 0$. From the elliptic regularity Theorem 1.4.6, u_0 is smooth. So is u_1.

Since P is essentially self-adjoint,

$$(\operatorname{Ker} P)^\perp = (\operatorname{Ker} \overline{P})^\perp = \operatorname{Im} \overline{P}^* = \operatorname{Im} \overline{P} = \overline{P}(\mathbf{H}^m(E)). \tag{2.1.51}$$

Thus there exists $v \in \mathbf{H}^m(E)$ such that $\overline{P}(v) = u_1$. From the elliptic regularity Theorem 1.4.6 again, v is smooth. So $\mathscr{C}^\infty(M, E) \subset \operatorname{Ker} P \oplus P(\mathscr{C}^\infty(M, E))$. The other direction is obvious.

The proof of Proposition 2.1.38 is completed. □

For more details of Hodge decomposition, please see my another note on Kähler Geometry.

Lemma 2.1.39. *For any $s \in \mathbb{R}$, there exists $C_s > 0$, such that for any $u \in \operatorname{Im} P$,*

$$\|Pu\|_s \geq C_s \|u\|_s. \tag{2.1.52}$$

Proof. We assume $s \geq 0$ first. If (2.1.52) does not hold, there exists $\{u_j\} \subset \operatorname{Im} P$ such that $\|u_j\|_s = 1$ and $\|Pu_j\|_s \to 0$. We may assume that $\|Pu_j\|_s \leq 1$ uniformly. From the elliptic estimates, $\|u_j\|_{s+m} \leq c(\|Pu_j\|_s + \|u_j\|_s) \leq 2c$. From Rellich theorem, there exists a subsequence of $\{u_j\}$, which we also denote by $\{u_j\}$, converges to $u \in (\operatorname{Ker} P)^\perp$ with respect to $\| \cdot \|_s$. Since P is closable, $Pu = 0$. So $u = 0$ and $\lim u_j = 0$ with respect to $\| \cdot \|$. Since $\|u_j\|_s$ converges, it must converge to $u = 0$, which is a contradiction with $\|u_j\|_s = 1$.

If $s < 0$, for any $u \in \operatorname{Im} P$, from (1.2.56) and (2.1.50),

$$\begin{aligned} \|u\|_s = \sup_{v \in \operatorname{Im} P} \frac{(u, v)}{\|v\|_{-s}} &= \sup_{w \in \operatorname{Im} P} \frac{(u, Pw)}{\|Pw\|_{-s}} = \sup_{w \in \operatorname{Im} P} \frac{(Pu, w)}{\|Pw\|_{-s}} \\ &\leq \sup_{w \in \operatorname{Im} P} \frac{\|Pu\|_s \|w\|_{-s}}{C_s \|w\|_{-s}} = C_s^{-1} \|Pu\|_s. \end{aligned} \tag{2.1.53}$$

The proof of Lemma 2.1.39 is completed. □

Let $P : \mathscr{C}^\infty(M,E) \to \mathscr{C}^\infty(M,E)$ be an essentially self-adjoint elliptic differential operator over a compact Riemannian manifold of order $m > 0$. From (2.1.50), $P : \mathrm{Im}P \to \mathrm{Im}P$ is an isomorphism. Then it has an inverse P^{-1}. Let $S : \mathscr{C}^\infty(M,E) \to \operatorname{Ker} P$ be the orthogonal decomposition with respect to the decomposition (2.1.50). Set

$$G := P^{-1} \circ (\mathrm{Id} - S) : \mathscr{C}^\infty(M,E) \to \mathscr{C}^\infty(M,E), \tag{2.1.54}$$

called **Green operator**. It is easy to see that

$$PG = GP = \mathrm{Id} - S. \tag{2.1.55}$$

Proposition 2.1.40. *For any $s \in \mathbb{R}$, there exists $C > 0$ such that for any $u \in \mathscr{C}^\infty(M,E)$,*

$$\|Gu\|_{s+m} \leq C\|u\|_s. \tag{2.1.56}$$

Then it extends a continuous map $G : \mathbf{H}^s(E) \to \mathbf{H}^{s+m}(E)$.

Proof. From Proposition 2.1.38, for any $u \in \mathscr{C}^\infty(M,E)$, there exists $u_0 \in \operatorname{Ker} P$, $u_1 \in \mathrm{Im}P$ such that $u = u_0 + u_1$. There exists $v_1 \in \mathrm{Im}P$ such that $Pv_1 = u_1$. From the elliptic estimate, Lemma 2.1.39 and (2.1.55),

$$\begin{aligned} \|Gu\|_{s+m} = \|GPv_1\|_{s+m} = \|v_1\|_{s+m} \leq c(\|Pv_1\|_s + \|v_1\|_s) \\ \leq c(1 + C_s^{-1})\|Pv_1\|_s = c(1 + C_s^{-1})\|u_1\|_s \leq c(1 + C_s^{-1})\|u\|_s. \end{aligned} \tag{2.1.57}$$

The proof of Proposition 2.1.40 is completed. □

2.1.6 Spectrum

Definition 2.1.41. Let T be a closed operator. Define

$$\begin{aligned} \rho(T) := \{\lambda \in \mathbb{C} : \operatorname{Ker}(\lambda \cdot \mathrm{Id} - T) = 0,\ \overline{\mathrm{Im}(\lambda \cdot \mathrm{Id} - T)} = \mathcal{H}, \\ (\lambda \cdot \mathrm{Id} - T)^{-1} \text{ is bounded}\}. \end{aligned} \tag{2.1.58}$$

It is called the **resolvent set** of T. The inverse $(\lambda \cdot \mathrm{Id} - T)^{-1}$ is called the resolvent of T at λ. The set

$$\sigma(T) := \mathbb{C} \backslash \rho(T) \tag{2.1.59}$$

is called the **spectrum** of T. If T is only closable, we define

$$\sigma(T) := \sigma(\overline{T}). \tag{2.1.60}$$

Definition 2.1.42. Let T be a closed operator. Define

$$\begin{aligned}\sigma_p(T) &:= \{\lambda \in \mathbb{C} : \mathrm{Ker}(\lambda \cdot \mathrm{Id} - T) \neq 0\};\\ \sigma_r(T) &:= \{\lambda \in \mathbb{C} : \mathrm{Ker}(\lambda \cdot \mathrm{Id} - T) = 0,\ \overline{\mathrm{Im}(\lambda \cdot \mathrm{Id} - T)} \neq \mathcal{H}\};\\ \sigma_c(T) &:= \{\lambda \in \mathbb{C} : \mathrm{Ker}(\lambda \cdot \mathrm{Id} - T) = 0,\ \overline{\mathrm{Im}(\lambda \cdot \mathrm{Id} - T)} = \mathcal{H},\\ &\qquad (\lambda \cdot \mathrm{Id} - T)^{-1} \text{ is unbounded}\},\end{aligned} \tag{2.1.61}$$

which are called **point spectrum, residual spectrum** and **continuous spectrum** respectively. Obviously,

$$\sigma(T) = \sigma_p(T) \sqcup \sigma_r(T) \sqcup \sigma_c(T). \tag{2.1.62}$$

Proposition 2.1.43. *If T is self-adjoint, then $\sigma(T) \subset \mathbb{R}$.*

Proof. It is easy to see that if we replace $\pm i\mu$ by $\lambda = v \pm i\mu \in \mathbb{C}$, Proposition 2.1.22 also holds. From (2.1.47), if $\mathrm{Im}\lambda \neq 0$, $\lambda \in \rho(T)$.

The proof of Proposition 2.1.43 is completed. □

Proposition 2.1.44. *If T is self-adjoint, then $\sigma_r(T) = \emptyset$.*

Proof. From Proposition 2.1.43, if $\lambda \in \sigma_r(T)$, $\lambda \in \mathbb{R}$. So $\mathrm{Ker}(\lambda \cdot \mathrm{Id} - T) = \mathrm{Im}(\lambda \cdot \mathrm{Id} - T)^{\perp}$.

The proof of Proposition 2.1.44 is completed. □

Definition 2.1.45. Let T be a self-adjoint operator. Define

$$\begin{aligned}\sigma_{ess}(T) &:= \sigma_c(T) \cup \{\lambda \in \sigma_p(T) : \dim \mathrm{Ker}(\lambda \cdot \mathrm{Id} - T) = +\infty\};\\ \sigma_d(T) &:= \{\lambda \in \sigma_p(T) : 0 < \dim \mathrm{Ker}(\lambda \cdot \mathrm{Id} - T) < +\infty\},\end{aligned} \tag{2.1.63}$$

which are called **essential spectrum** and **discrete spectrum** respectively. The element of $\sigma_d(T)$ is called an eigenvalue of T. Obviously,

$$\sigma(T) = \sigma_{ess}(T) \sqcup \sigma_d(T). \tag{2.1.64}$$

Theorem 2.1.46 (Spectral theorem for compact operators). [2] *Let $\mathcal{H}$ be a infinite dimensional Hilbert space. Let B be a compact self-adjoint operator on $\mathcal{H}$. Then there exists a complete orthonormal basis $\{\varphi_j\}_{j=1}^{\infty}$ of $\mathcal{H}$ and a real sequence $\{\lambda_1, \cdots, \lambda_n, \cdots\}$ (at most countable) such that $B\varphi_j = \lambda_j\varphi_j$ and $\lim_{j\to+\infty} \lambda_j = 0$ (if the real sequence is not discrete).*

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[2] Theorem 4.4.7 in ”Functional analysis I” by Zhang Gongqing

Theorem 2.1.47. *Let* $P : \mathscr{C}^\infty(M, E) \to \mathscr{C}^\infty(M, E)$ *be an essentially self-adjoint elliptic differential operator over a compact Riemannian manifold of order* $m > 0$*. Then*

(1) for $\lambda \in \sigma_d(P)$*, the* λ*-eigenspace* $E_\lambda = \operatorname{Ker}(\lambda \operatorname{Id} - P)$ *is finite dimensional and consist of smooth sections;*

(2) there is a direct sum decomposition

$$L^2(M, E) = \bigoplus_{\lambda \in \sigma_d(P)} E_\lambda. \tag{2.1.65}$$

(3) the spectrum

$$\sigma(P) = \sigma_d(P) \tag{2.1.66}$$

and it is discrete.

Proof. (1). Since $\lambda \operatorname{Id} - P$ is elliptic, (1) follows from Theorem 1.4.6 (3).

(2). Let E_s be the closure of $\operatorname{Im} P$ with respect to $\|\cdot\|_s$. Then $L^2(M, E) = \operatorname{Ker} P \oplus E_0$ and from the Rellich theorem and Proposition 2.1.46, the Green operator

$$G : E_0 \to E_m \hookrightarrow E_0 \tag{2.1.67}$$

is a compact self-adjoint operator. From Theorem 2.1.46, since $\ker G = 0$, there exists a complete orthonormal basis $\{\varphi_j\}_{j=1}^\infty$ of E_0 and a real sequence $\{\lambda_1^{-1}, \cdots, \lambda_n^{-1}, \cdots\}$ such that $G\varphi_j = \lambda_j^{-1}\varphi_j$ and $\lim_{j\to+\infty} \lambda_j^{-1} = 0$. So we get (2).

(3) From Proposition 2.1.38, $\mathscr{C}^\infty(M, E) = \operatorname{Ker}(P - \lambda \operatorname{Id}) \oplus \operatorname{Im}(P - \lambda \operatorname{Id})$. If $\operatorname{Ker}(P - \lambda \operatorname{Id}) = 0$, $\mathscr{C}^\infty(M, E) = \operatorname{Im}(P - \lambda \operatorname{Id})$. From Lemma 2.1.39, there exists $C > 0$, such that for any $u \in \mathscr{C}^\infty(M, E)$, $\|(P - \lambda \operatorname{Id})u\| \geq C\|u\|$. So for any $u \in \mathscr{C}^\infty(M, E)$, $\|(P - \lambda \operatorname{Id})^{-1}u\| \leq C^{-1}\|u\|$. So $\sigma_c(P) = \emptyset$.

The proof of Theorem 2.1.47 is completed.

□